Measure Theory with Ergodic Horizons
Lecture 3

Premeasures.

To define interesting nonatomic measures on σ-algebras, e.g. the Borel σ-algebra of metric spaces, we first define countably additive functions on algebras and then extend them to the generated σ-algebras. A ctbly additive function $m : \mathcal{A} \to [0,\infty]$ on an algebra $\mathcal{A}$ with $m(\emptyset) = 0$ is called a premeasure to emphasize that $\mathcal{A}$ doesn't need to be a σ-algebra.

Bernoulli premeasure on the cylinder algebra of $2^{\mathbb{N}}$.

Let $p \in (0,1)$ and let $\mathcal{C}$ denote the algebra on $2^{\mathbb{N}}$ generated by the cylinders, i.e. sets of the form $[w]$ for some finite word $w \in 2^{<\mathbb{N}}$. Note that sets in $\mathcal{C}$ are exactly the finite disjoint unions of cylinders. We define the Bernoulli(p) premeasure on $\mathcal{C}$ by firstly defining its values on cylinders:

$$\tilde{\mu}_p([01100]) := p^2 \cdot (1-p)^3$$

$$\tilde{\mu}_p([w]) := p^{(\# \text{ of } 1\text{s in } w)} (1-p)^{(\# \text{ of } 0\text{s in } w)}.$$

We can extend this to a function on $\mathcal{C}$ by setting

$$\mu_p(C) := \sum_{D \in \mathcal{P}} \tilde{\mu}_p(D), \text{ where } \mathcal{P} \text{ is some finite partition of } C \text{ into cylinders.}$$

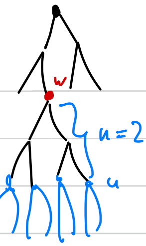
This definition works, but we need to show its correctness as well as ctbl additivity. We show correctness (i.e. independence of representation of each element of $\mathcal{C}$ as a finite disjoint union of cylinders) and finite additivity together in the following two claims.

Claim (a). $\tilde{\mu}_p$ is finitely additive on equal-length cylinders, i.e. for any cylinder $[w]$ and $n \in \mathbb{N}$,

$$\tilde{\mu}_p([w]) = \sum_{u \in 2^n} \tilde{\mu}_p([wu]).$$

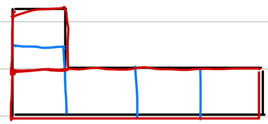
Proof. By induction on n, it's enough to check the $n=1$ case:

$$\tilde{\mu}_p([w0]) + \tilde{\mu}_p([w1]) = \tilde{\mu}_p[w]\cdot(1-p) + \tilde{\mu}_p[w]\cdot p = \tilde{\mu}_p([w]). \qquad \square$$



Claim (b). Let $C \in \mathcal{C}$ and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be two partitions of C into cylinders. Then

$$\sum_{Q_1 \in \mathcal{Q}_1} \tilde{\mu}_p(Q_1) = \sum_{Q_2 \in \mathcal{Q}_2} \tilde{\mu}_p(Q_2).$$



Proof. Let $\mathcal{R}$ be a common refinement of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, and refining it further, we may assume that all cylinders in $\mathcal{R}$ have the same length, i.e. for some $n \in \mathbb{N}$, all cylinders in $\mathcal{R}$ are of the form $[w]$ where $w \in 2^n$.

Then,

$$\sum_{Q_1 \in \mathcal{Q}_1} \tilde{\mu}_p(Q_1) \overset{\text{Claim (a)}}{=} \sum_{Q_1 \in \mathcal{Q}_1} \sum_{\substack{R \in \mathcal{R} \\ R \subseteq Q_1}} \tilde{\mu}_p(R) = \sum_{R \in \mathcal{R}} \tilde{\mu}_p(R) \overset{\text{Claim (a)}}{=} \sum_{Q_2 \in \mathcal{Q}_2} \sum_{\substack{R \in \mathcal{R} \\ R \subseteq Q_2}} \tilde{\mu}_p(R) = \sum_{Q_2 \in \mathcal{Q}_2} \tilde{\mu}_p(Q_2). \qquad \square$$

Claim (b) implies that μ_p is well-defined and also that it is finitely additive.

Claim (c). μ_p is automatically ctbly additive because there aren't any nontrivial infinite partitions of elements of $\mathcal{C}$ into elements of $\mathcal{C}$.

Proof. Suppose $C = \bigsqcup_{n \in \mathbb{N}} C_n$ is a partition of $C \in \mathcal{C}$ and each $C_n \in \mathcal{C}$. Then C and the C_n are clopen, hence C is compact and C_n are open, so the C_n form an open cover of C, hence there is a finite subcover, but the pairwise disjointness of the C_n implies that those C_n not in this finite cover, must be empty. $\square$

Lebesgue premeasure on $\mathbb{R}^d$.

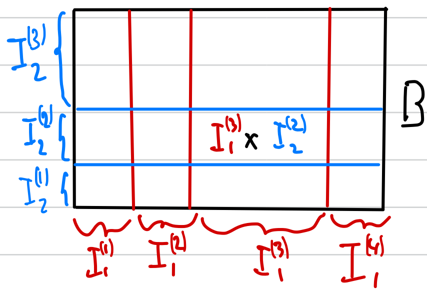
We define this analogously to the Bernoulli premeasure, but instead of cylinders, we use boxes, i.e. sets of the form $I_1 \times I_2 \times \ldots \times I_d$, where each I_k is an interval in $\mathbb{R}$, possibly unbounded, e.g. $(1,\infty)$ or $[0,1)$. Let $\mathcal{A}$ denote the algebra generated by boxes, i.e. the collection of finite disjoint unions of boxes.
We first define the premeasure on boxes:

$$\tilde{\lambda}(I_1 \times I_2 \times \ldots \times I_d) := \mathrm{lh}(I_1) \cdot \mathrm{lh}(I_2) \cdot \ldots \cdot \mathrm{lh}(I_d),$$

where $\mathrm{lh}(\text{interval}) := (\text{right endpoint}) - (\text{left endpoint})$ and $0 \cdot \infty := 0$.
Now define the potential premeasure λ on $\mathcal{A}$ by: for $A \in \mathcal{A}$,

$$\lambda(A) := \sum_{B \in \mathcal{P}} \tilde{\lambda}(B), \text{ for some finite partition } \mathcal{P} \text{ of } A \text{ into boxes.}$$

Again we need to prove the correctness of this definition (i.e. that it doesn't depend on the choice of the partition $\mathcal{P}$), as well as ctbl additivity. We again prove correctness and finite additivity together in Claims (a) and (b) below.
In Claim (a), the role of equal-length cylinders will be played by grid-partitions, namely a grid-partition of a box $B := I_1 \times I_2 \times \ldots \times I_d$ is a finite partition $\mathcal{P}$ of B into boxes of the following form: each I_k is partitioned into intervals $I_k = \bigsqcup_{n < N_k} I_k^{(n)}$ and $\mathcal{P} = \{ I_1^{n_1} \times I_2^{n_2} \times \ldots \times I_d^{n_d} : (n_1, n_2, \ldots, n_d) \in N_1 \times N_2 \times \ldots \times N_d \}$, where we view $N := \{0,1,2,\ldots,N-1\}$.



Claim (a). If $\mathcal{P}$ is a grid-partition of a box B, then $\tilde{\lambda}(B) = \sum_{P \in \mathcal{P}} \tilde{\lambda}(P)$.

Proof. This hold trivially for $d=1$, i.e. for intervals, and for general d, it follows by induction on d using distributivity law:

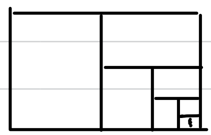
$$(a_1+a_2+\ldots+a_k)\cdot(b_1+b_2+\ldots+b_\ell)=\sum_{\substack{i\le k\\ j\le \ell}} a_i b_j . \qquad \square$$

Claim (b). Let $\mathcal{Q}_1, \mathcal{Q}_2$ be two partitions of an $A \in \mathcal{A}$ into boxes. Then

$$\sum_{Q_1\in\mathcal{Q}_1} \tilde{\lambda}(Q_1) = \sum_{Q_2\in\mathcal{Q}_2} \tilde{\lambda}(Q_2).$$

Proof. Same as with Bernoulli, but using grid-partitions, HW.

Claim (b) implies that λ is well-defined on $\mathcal{A}$ and that it is also finitely additive. Unlike the Bernoulli premeasure, λ on $\mathcal{A}$ is not automatically ctbly additive, it is possible to have a nontrivial ∞finite partition of a box into boxes:



Also, $\mathbb{R}^d$ is not compact and boxes are not clopen.

Claim (c). λ is ctbly additive.

Proof. We only the special case when a bdd box B is partitioned into infinitely many boxes: $B=\bigsqcup_{n\in\mathbb{N}} B_n$. The general case follows from this special case and is left as HW.

So, we suppose that B is bdd. In the case of Bernoulli, we used that B was compact and the B_n were open, while in our case neither may be true. However, we can replace B with a closed box $B' \subseteq B$ with $\lambda(B\setminus B') < \varepsilon/2$ for some apriori fixed arbitrary $\varepsilon>0$. Similarly, we can replace B_n with an open box $\tilde{B}_n \supseteq B_n$ with $\lambda(\tilde{B}_n \setminus B_n) < \varepsilon/_{2\cdot 2^{n+1}}$. Thus $\{\hat{B}_n\}_{n\in\mathbb{N}}$ is an open cover of the compact set B', hence there is a finite subcover, so we can use finite additivity...